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Asymptotic Behavior for the Liouville Equations

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A new proof of the diffusion approximation for ordinary differential equations is given. It is based on an asymptotic expansion of the solution of the corresponding Liouville partial differential equations. In contrast to previous results obtained for the suspension under Holderian mappings of subshift of finite type or Fourier analysis techniques, our proof relies only on symbolic dynamics.

KEY WORDS: Diffusion process; geodesic flow; continued fractions; expanding maps.

1. INTRODUCTION

The subject of this paper is to study the behaviour of the solution of the Liouville equations. We give a new proof for the diffusion equation using vector fields which are more singular than those considered in the original proof.[4] Our motivation is twofold. First, we shall discuss a class of problems which is not more general, but rather disjoint from the one considered by Arnold,[2] Bardos *et al.*[4] Secondly, we will obtain the diffusion equation as the limit of the reversible kinetic equation scaled appropriately. Actually, ref. 4 deals with a result on reversible models of transport equations. A diffusion obtained through a convenient scaling from a reversible kinetic equation is produced by the collisions of the particles with the boundary, parametrized by $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. These particles are assumed to reflect according reversibility law which induces convenient mixing properties (see also ref. 7). Optimal convergence results are simply obtained and this is made possible because the model, using the transformation T of

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hyperbolic automorphism of the torus—Arnold's cat map—can be handled with Fourier decomposition instead of coding with Markov Partition.

Here we place our result in another context. We do not assume that the domain is a torus, but instead that is a Riemann surface. In fact, we consider a class of mixing flows whose autocorrelation functions decay sufficiently fast, or alternately, we consider the class of transitive Anosov flows on a Riemann surface $\mathscr{M}$. We deal with a system of ordinary differential equations (ODEs) of the form

$$\partial_t X = \varepsilon f(Y), \qquad \partial_t Y = \omega(Y) \tag{1.1}$$

where the variable X belongs to $\mathbb{R}^n$ and Y to the non-compact n-dimensional manifold $\mathscr{M}$. The variable X plays the role of a "slow" variable and Y the "fast" one. The maps $\omega, f\colon \mathscr{M} \to \mathbb{R}^n$ are given vector fields; we assume that ω is independent of the slow variables ($\omega = \omega(Y)$), and we strengthen the uniform distribution hypothesis from ergodicity to mixing. Here $\varepsilon \geqslant 0$ is an infinitesimal parameter. The Liouville partial differential equations (PDEs) corresponding to Eqs. (1.1) are

$$\partial_t F = \varepsilon f(y) \cdot \partial_x F + \omega(y) \cdot \partial_y F, \qquad F \equiv F_\varepsilon(t, x, y) \qquad F|_{t=0} = h(x) \tag{1.2}$$

The solution of Eq. (1.2) is given by the formula:

$$F(t, x, y) = F(X_t(x, y)) \tag{1.3}$$

Using the formula (1.3), and functional analysis we get quantitative informations from Eqs. (1.1). The interest to study the Liouville equation of a vector fields to obtain the information on its trajectory flow arises from the fact that the weak convergence of the solution of Liouville equations implies the convergence of the flow of the associated system almost everywhere.

The difference between ref. 4 and our approach is that here we place ourselves on $\mathscr{M} = \mathbb{H}/SL(2, \mathbb{Z})$, the Riemann surface with constant curvature, which is non-compact but with finite volume and preserved Gauss measure. The Arnold's cat map is replaced by the geodesic flow coded by the transformation of continued fractions. Therefore we use elementary results from number theory about continued fractions. We discuss the approach for Perron–Frobenius or the transfer operator method for subshifts of finite type. The main step for applying the transfer operator method in this case is Arnoux[3] and Series' construction[14] of symbolic dynamics for these flows, reducing this method to the dynamics to special flows over "analytic expanding maps" of the unit interval. For the modular surface this map is just Gauss'continued fraction transformation. Another difference between ref. 4 and our approach is that, for proving decorrelation property, ref. 4 uses Fourier analysis, while we need to connect the geodesic flow to the special associated flow. An essential point in our paper is that we have not assumed the suspension g of the special flow to be bounded, but only to belong to the space L^1. A closely related result proved under less restrictive hypotheses may be found in ref. 13.

2. SETUP AND STATEMENT OF RESULT

Let $\mathscr{M}$ be a smooth non-compact manifold of dimension n, equipped with a smooth density $|dY|$ normalized to satisfy the relation $\int_{\mathscr{M}} |dY| = 1$. For a small $\varepsilon \geqslant 0$, we consider the system of ODEs on $\mathbb{R}^n \times \mathscr{M}$ given by

$$\frac{dX}{dt} = \varepsilon f(Y) \tag{2.1}$$

and

$$\frac{dY}{dt} = \omega(Y) \tag{2.2}$$

which are associated to the perturbation $(\varepsilon f, \omega)$ of the vector field $(0, \omega)$. Here, X and Y denote any elements of $\mathbb{R}^n$ and $\mathscr{M}$, respectively. We shall impose the following assumptions:

(H1) $f \in \mathscr{C}^1(\mathscr{M})$ and $\omega \in \mathscr{C}^1(\mathscr{M})$;

(H2) the density $|dY|$ is invariant under the flow generated by ω.

By the standard ODE theory, it is well-known that, under hypotheses (H1) and (H2), the Eqs. (2.1)–(2.2) generate global flows which leave the respective domains $D_{\mathbb{R}} \equiv \bar{B}_{\mathbb{R}} \times \mathscr{M}$ and $\bar{B}_{\mathbb{R}}$ invariant (here $\bar{B}_{\mathbb{R}}$ denotes the closed ball of radius R, centered on the origin in $\mathbb{R}^n$). We shall denote these flows by

$$(X_t^\varepsilon, Y_t) \equiv (X_t^\varepsilon, Y_t)(x, y) \tag{2.3}$$

where

$$X_t^\varepsilon(x, y) = x + \varepsilon \int_0^t f(Y_s(y))\, ds \tag{2.4}$$

The notation here means that, $t \mapsto (X_t^\varepsilon, Y_t) \equiv (X_t^\varepsilon, Y_t)(x, y)$ is the integral curve of the vector field $(\varepsilon f, \omega)$ passing through the initial condition (x, y)

at the time $t=0$. Furthermore, It is worth to point out that the unperturbed flow is consistently denoted (X_t^0, Y_t). The Liouville equation corresponding to Eqs. (2.1)–(2.2) is then given by

$$\partial_t \varphi = \varepsilon f(y) \cdot \partial_x \varphi + \omega(y) \cdot \partial_y \varphi, \qquad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n, \quad y \in \mathcal{M} \tag{2.5}$$

with the initial condition

$$\varphi(0, x) = \phi(x), \qquad x \in \mathbb{R}^n \tag{2.6}$$

If a small parameter $\varepsilon > 0$ is introduced so that t is changed into t/ε. Then problem of interest becomes

$$\varepsilon^2\, \partial_t \varphi_\varepsilon = \varepsilon f(y) \cdot \partial_x \varphi_\varepsilon + \omega(y) \cdot \partial_y \varphi_\varepsilon, \qquad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n, \quad y \in \mathcal{M} \tag{2.7}$$

$$\varphi_\varepsilon \equiv \varphi_\varepsilon(t, x, y) \qquad \varphi_\varepsilon|_{t=0} = \phi(x), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n, \quad y \in \mathcal{M} \tag{2.8}$$

Now, integrating these equations by the characteristic method, we get:

$$\varphi_\varepsilon \equiv \varphi_\varepsilon(t, x, y) = \phi(X_{t/\varepsilon^2}(x, y)) = \phi\left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s(y))\, ds \right) + O(\varepsilon) \tag{2.9}$$

Therefore, most of the analysis is reduced to studying the limit, as $\varepsilon \to 0$, of the expression

$$\psi_\varepsilon(t, x, y) = \phi\left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s(y))\, ds \right) \tag{2.10}$$

It will be convenient to introduce the following notation: let r and $s \in \mathbb{N}^* = \mathbb{N} - \{0\}$; if A and B are two vectors in $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively, define the $r \times s$ matrix $A \otimes B := A^t \cdot B$, and denote by $A^{\otimes 2}$ the $r \times r$ symmetric matrix $A \otimes A$. Let M and N be two $r \times s$ matrices and define the real number $M : N = N : M = \sum_{i,j} M_{i,j} N_{i,j}$. It is an inner product on $\mathcal{M}_n$. With these notation, if X is a $r \times s$ matrix, A and B in $\mathbb{R}^r$, one has

$$X\colon A \otimes B = \sum_{i,j} X_{i,j} A_i B_j = {}^t(XB)\, A = \langle A, XB \rangle$$

In what follows, the notation $\langle \cdot \rangle$ stands for $\langle F \rangle = \int_{\mathcal{M}} F(Y)\, |dY|$.

The main result of this paper can now be stated as follows:

Theorem 2.1. Let $f\colon \mathcal{M} \to \mathbb{R}^n$ be in the class $\mathcal{C}^3(\mathcal{M})$ with mean value $\langle f \rangle = 0$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ be an initial data. Then, there exists a positive matrix, denoted κ^2, limit of the sequence of matrices $(\langle((1/\sqrt{T})\times \int_0^T f(Y_s)\,ds)^{\otimes 2}\rangle)_{T\geqslant 1}$ such that, if $u(t,x)\in\mathscr{C}^{1,2}(\mathbb{R}^+\times\mathbb{R}^n)$ denotes the unique solution of the heat equation

$$\partial_t u=\nabla_x\cdot(\kappa^2\nabla_x u);\qquad u(0,x)=\phi(x) \tag{2.11}$$

the function φ_ε defined by Eq. (2.9), converges to $u(t,x)$ as $\varepsilon\to 0$ in the following sense: for any $\tau>0$ and any compact $K\subset\mathbb{R}^n_x$

$$\begin{aligned}\langle\varphi_\varepsilon(t,x,y)\rangle &\longrightarrow u(t,x), \qquad C^0([0,\tau]\times K\times\mathscr{M});\\ \varphi_\varepsilon(t,x,y) &\longrightarrow u(t,x), \qquad C^0([0,\tau],w^*-L^\infty(\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M}))\end{aligned}$$

Furthermore, with $\psi_\varepsilon(t,x,y)$ defined by Eq. (2.10) we have:

$$\|\varphi_\varepsilon-\psi_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^n\times\mathscr{M})}=O(\varepsilon) \tag{2.12}$$

2.1. Approximation by Diffusion. Principle of the Proof

The proof of our theorem is inspired by the proof of the Itô formula for Brownian motions (cf. for instance ref. 5). Since $\phi\in\mathscr{C}^3_b$, the starting point is the Taylor's formula with "remainder" at the order two for the increment:

$$\begin{aligned}&\langle\psi_\varepsilon(t+\tau,x,\cdot)\rangle-\langle\psi_\varepsilon(t,x,\cdot)\rangle\\ &\quad=\left\langle\nabla_x\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2}f(Y_s(y))\,ds\right)\cdot\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2}f(Y_s(y))\,ds\right\rangle\\ &\qquad+\tfrac{1}{2}\left\langle\nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2}f(Y_s(y))\,ds\right):\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2}f\Big(Y_s(y)\Big)\,ds\right)^{\otimes 2}\right\rangle\\ &\qquad+O\left(\left\langle\left|\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2}f\Big(Y_s(y)\Big)\,ds\right)\right|^3\right\rangle\right)\end{aligned} \tag{2.13}$$

for all $\tau>0$.

The analysis of the limit of Eq. (2.13) as $\varepsilon\to 0$ in the above expression will be done in four steps under some hypotheses:

- First prove that

$$\begin{aligned}&\left|\left\langle\nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2}f(Y_s(y))\,ds\right):\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2}f(Y_s(y))\,ds\right)^{\otimes 2}\right\rangle\right.\\ &\left.-\left\langle\nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2}f(Y_s(y))\,ds\right)\right\rangle:\left\langle\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2}f(Y_s(y))\,ds\right)^{\otimes 2}\right\rangle\right|=o(\tau)\end{aligned} \tag{2.14}$$

when $\varepsilon \to 0$ for $0 < t < t + \tau$. By the definition of φ_ε,

$$\left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s(y))\, ds \right) \right\rangle = \nabla_x^2 \langle \varphi_\varepsilon(t, x, \cdot) \rangle \tag{2.15}$$

- Secondly prove that the remainder

$$O\left(\left| \left(\varepsilon \int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s(y))\, ds \right) \right|^3 \right) = o(\tau), \qquad \text{as} \quad \varepsilon \to 0 \tag{2.16}$$

- Thirdly show that the family

$$\left\langle \left(\varepsilon \int_0^{t/\varepsilon^2} f(Y_s(y))\, ds \right)^{\otimes 2} \right\rangle \longrightarrow \kappa^2, \qquad \text{as} \quad \varepsilon \to 0 \tag{2.17}$$

where κ^2 is the diffusion coefficient.

- Finally, under Eqs. (2.14)–(2.16)–(2.17) letting $\varepsilon \to 0$ in Eq. (2.13), get

$$\langle \psi_\varepsilon(t + \tau, x, \cdot) \rangle - \langle \psi_\varepsilon(t, x, \cdot) \rangle = \tfrac{1}{2} \tau \kappa^2 \colon \nabla_x^2 u(t, x) + O(\tau)^{1/2} \tag{2.18}$$

Now, there are many ways to proceed:

- We can apply the result of M. Ratner,[15] or Dumas–Golse,[8] assuming that Y_t is an ergodic Anosov flow on $\mathscr{M}$ with invariant continuous measure, and to show that it satisfies the Central Limit Theorem.

- We can use probabilistic method, the invariance principle of Donsker (Tightness).

- Our proof relies entirely on functional analysis, using a priori estimates of ODEs and symbolic dynamics, while other authors methods depend on probabilistic methods. The essential point of the proof is based on the decorrelation of two intervals of time, uniformly with respect to their size, under the hypothesis only that their distance is large enough; we prove that the average of the different products which appear in the Taylor formula are, in the limit completely decorrelated and therefore converge to the product of the corresponding limiting averages. In the original Itô formula, this point is straightforward, because the Brownian motion is by hypothesis a process with independent increments. By contrast in the present paper, the independence can only be obtained in the limit as $\varepsilon \to 0$ and, as will be shown below, it is a consequence of the different mixing properties inherited from the mapping Y_t. In the ref. 4, these properties has being obtained by using elementary techniques for Fourier series expansion. Here, we are in the case where the flow Y_t viewed as a geodesic flow on non-compact but with finite volume Riemann surface, is coded by continued fractions. Therefore to have mixing properties, the Perron-Frobenuis operator (or as we prefer to call it the transfer operator) appears naturally in connection with the subshifts of finite type.

The outline of the paper is as follows: Section 3 deals with symbolic dynamics. We connect the flow Y_t with the continued fractions. We give the relationship between the flow Y_t and the special flow associated. In Section 4 we prove the decorrelation property and the main theorem is established. The spectral properties of the Perron–Frobenius operator is presented in the Appendix.

3. SYMBOLIC DYNAMICS

Following an idea of Arnoux[3] or Series,[14] which has the advantage of being simple and more easily to adapt to our problem, we switch to the symbolic setting.

3.1. On Coding of the Flow Y_t with Continued Fractions

We shall be interested in the hyperbolic geometry model $\mathscr{M} = \mathbb{H}/\Gamma$, where $\mathbb{H}$ denotes the Poincaré upper half plane defined by $\{x + iy : y > 0\}$ with the metric and $ds^2 = (dx^2 + dy^2)/y^2$, and $\Gamma = SL(2, \mathbb{Z})$ (the 2×2 matrices with integer coefficients having determinant 1). Geodesics are arcs of a semicircle centered on the real axis $\mathbb{R} = \{z \in \mathbb{C} : \mathscr{I}m\, z = 0\}$ or a part of a vertical line perpendicular to $\mathbb{R}$. The full modular group $SL(2, \mathbb{Z})$ acts on $\mathbb{H}$ by isometries of $\mathbb{H}$ that is mapping $((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \cdot z) \mapsto (az + b)/(cz + d)$. $\mathbb{H}/SL(2, \mathbb{Z})$ is a Riemann surface with constant curvature -1; it is non-compact but has finite volume. Geodesics on this surface are projections of geodesics in the universal cover $\mathbb{H}/SL(2, \mathbb{Z})$ of $\mathbb{H}/SL(2, \mathbb{Z})$ and the geodesic flow is a flow on the unit tangent bundle.

In order to set up symbolic dynamics for the geodesic on the modular surface $\mathscr{M}$ we need to investigate the connection between the dynamics of Y_t on $\mathscr{M} = \mathbb{H}/SL(2, \mathbb{Z})$ and the continued fraction transformation. This connection can be observed in the following manner: any real number $x \in\,]0, 1[$ can be expanded as the form:

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [0; a_1, a_2,..., a_n,...], \qquad (a_k > 0,\ \forall k \geqslant 1) \tag{3.1}$$

In the particular case, when the sequence (a_k) is periodic, denote

$$[0; \overline{a_1, a_2, \ldots, a_n}] = [0; a_1, \ldots, a_n, a_1, \ldots, a_n, a_1, \ldots]$$

An immediate computation shows that $a_1 = [1/x]$, where $[y]$ means the largest integer $n \leqslant y$; if we put $x_1 = \{1/x\}$ where $\{t\} = t - [t]$ fractional part of t, we have $a_2 = [1/x_1]$.

More generally, define the following mappings

$$g\colon]0, 1[\longrightarrow \mathbb{N} \qquad \text{s.t.} \quad g\colon x \longrightarrow \left[\frac{1}{x}\right]$$

$$T\colon [0, 1[\longrightarrow [0, 1[\qquad \text{s.t.} \quad T\colon x \longrightarrow \left\{\frac{1}{x}\right\} \qquad \text{if} \quad x \neq 0, \quad \text{and} \quad T(0) = 0$$

We verify at once that $a_n = g(T^{n-1}(x))$, which is well-defined whenever $T^{n-1}(x) \neq 0$.

The transformation T is called the continued fraction transformation. It admits a geometric extension $\tilde{T}$ defined on a subset of $\mathbb{R}^2$, in which a way that if π denotes the projection on the first variable, $\pi(\tilde{T}(x, y)) = T(x)$, which is defined on the subset Σ of the plane of equations $0 \leqslant x < 1$ and $0 \leqslant y < 1/(x+1)$ if $x < 1/2$, $0 < y < 1/(x+1)$ otherwise and is given by

$$\tilde{T}\colon \Sigma \to \Sigma$$

$$\tilde{T}\colon (x, y) \mapsto \left(\left\{\frac{1}{x}\right\}, x - x^2 y\right) \qquad \text{if} \quad x \neq 0,$$

$$(0, x) \mapsto (0, 0)$$

This map is one-to-one except at the points $(0, y)$ which have the same image and the points $(x, 0)$ which have an empty preimage if $x \neq 0$. The map $\tilde{T}$ sends vertical segments to vertical segments and is discontinuous on the set $x = 1/n$, $n \in \mathbb{N}$, but continuous on the curve lying in the rectangle which bound it. This map can be represented in the following manner: let $\tilde{g}$ be the number of rectangles which contain the point (x, y) defined by

$$\tilde{g}\colon \Sigma \to \mathbb{N}; \qquad (x, y) \mapsto \left[\frac{1}{x}\right] \qquad \text{if} \quad x \neq 0$$

Put

$$\Sigma' = \Sigma \Big\backslash \left(\{(0, 0)\} \cup \bigcup_{k > 0} \tilde{T}^{-k}(0, 0) \cup \bigcup_{k \geqslant 0} \tilde{T}^k(x, 0)\right)$$

Let $\mathbb{N}^{*\mathbb{Z}}$ denote the set of all sequences $(a_n)_{n\in\mathbb{Z}}$ with entries in $\mathbb{N}$, having "compact support," i.e., $\{n\in\mathbb{Z} \mid a_n \neq 0\}$ is finite. There exists a map

$$h\colon \Sigma' \longrightarrow \mathbb{N}^{*\mathbb{Z}}$$
$$(x, y) \overset{\sim}{\longmapsto} (\tilde{g}(\tilde{T}^{n-1}(x, y)))_{n\in\mathbb{Z}}$$

such that

$$\phi \circ h = h \circ \tilde{T}$$

where ϕ is the shift operator defined in (3.2) below. This formula proves that the coding map which to the point (x, y) associate $(a_n)_{n\in\mathbb{Z}}$ conjugates $\tilde{T}$ to the shift operator ϕ on $\mathbb{N}^{*\mathbb{Z}}$.

3.2. Relation Between the Geodesic Flow Y_t and the Associated Special Flow S_t

Let $\mathcal{O}$ be the space of doubly infinite sequences of positive integers. We first introduce the suspension of $\tilde{\mathcal{O}}$ by g, defined as the quotient space

$$\tilde{\mathcal{O}} = \{(z, s) \in \mathcal{O} \times \mathbb{R}^+ \mid 0 \leqslant s \leqslant g(z)\}/\sim$$

where the equivalence $\sim$ is defined by $(z, g(z)) \sim (\phi z, 0)$, and ϕ is the shift operator on $\mathcal{O}$ defined by $(\phi z)_i = z_{i-1}$ for all $i\in\mathbb{Z}$. Moreover g is a positive continuous function defined on $\mathcal{O}$. This allows us to construct the "special flow" S_t: The special flow over $\mathcal{O}$ with the first return map T and the recurrence time g, is the flow S_t defined for $t < \inf_{z\in\mathcal{O}} g(z)$ by

$$\begin{aligned} S_t(z, s) &= (z, s+t), && \text{if} \quad s+t < g(z), \\ S_t(z, s) &= (\phi z, s+t-g(z)), && \text{if} \quad s+t \geqslant g(z) \end{aligned}$$

Let μ denotes a ϕ-invariant probability measure on $\mathcal{O}$ such that $dv = (d\mu(z) \times ds)/\bar{g}$ where $\bar{g} = \int_{\mathcal{O}} g(z)\, d\mu(s)$, according to the following scheme

$$\chi\colon \mathcal{M} \to \tilde{\mathcal{O}}$$
$$Y_t \mapsto S_t$$
$$\mu \mapsto \tilde{v} = dv(z)\, ds$$

The relation connecting the flow Y_t to its special flow S_t is given by

$$S_t(z, s) = \left(T^n z, \sigma + s - \sum_{r=0}^{n-1} g(T^r z) \right) \tag{3.2}$$

where n satisfies

$$\sum_{r=0}^{n-1} g(T^r z) \leqslant \sigma + s < \sum_{r=0}^{n} g(T^r z) \tag{3.3}$$

The map which sends (z, y) to the sequence $(a_n)_{n \in \mathbb{Z}}$, where (a_n) is the coding of (x, y) defined in the previous paragraph, conjugate the geodesic flow to a special flow S_t over $\mathbb{N}^{*\mathbb{Z}} \times \{0, 1\}$ with the return time $-2 \ln[0; a_1, a_2,..., a_n,...]$. The first return map of the special flow S_t is a two-fold covering of the shift.

3.3. Perron–Frobenius Operator

For the transformation $T: I \to I$ defined by

$$Tx = \begin{cases} 1/x & \text{mod.1} \quad x \neq 0 \\ 0 & \qquad\quad\; x = 0 \end{cases}$$

where $I = [0, 1]$, direct computation yields $T^k[a_1, a_2,...] = [a_{k+1},...]$, for $k = 0, 1, 2,...,$ and $a_k = [(T^{k-1}x)^{-1}]$. Therefore, the distribution of the entries a_k in the continued fraction expansion of x is closely related to the ergodic properties of the dynamical system T. Obviously, the Gauss map T is an analytic *expanding Markov map*; that is, there exists a countable partition $\mathscr{A} = \{I_i\}_{i \in \mathscr{F}}$ of I (where $\mathscr{F}$ is a countable indexing set) into non-trivial intervals $I_i = [t_{i-1}, t_i]$ such that:

(i) $I = \bigcup_{i \in \mathscr{F}} I_i$;

(ii) $\operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset$, if $i \neq j$;

(iii) $T_i := T|_{I_i}$ is monotone and of class $\mathscr{C}^k$ for some $k \geqslant 1$;

(iv) $|(T^n)'(x)| \geqslant \delta > 1$ for some $n \geqslant 1$ and all $x \in I$.

Now, for the partition $\mathscr{A} = \{I_n\}_{n \in \mathbb{N}}$ with $I_n = [1/(n+1), 1/n]$, we find out that $T_{|_{I_n}}(x) = T_n(x) = (1/x) - n$ is analytic in $x \neq 0$, and $|(T^2)'(x)| \geqslant 4 > 1$ for all $x \in I$. Furthermore we get for all $n \in \mathbb{N}$: $TI_n = I$, so $\mathbf{1}_{TI_n} \equiv 1$ for all $n \in \mathbb{N}$. The inverse maps have the explicit form

$$\psi_i = T_i^{-1}: I \to I_i \qquad \psi_i(x) = \frac{1}{x+i}$$

and hence are meromorphic in the entire z-plane with a simple pole at $z = -i$.

Since $T\mathscr{S}_{\mathscr{A}} = \{0, 1\}$ and hence $T\mathscr{S}_{\mathscr{P}} = \mathscr{S}_{\mathscr{P}}$ if $\mathscr{S}_{\mathscr{P}} = \{0, 1\}$, the partition $\mathscr{P}$ is the trivial partition $\mathscr{P} = \{I\}$. We introduce now the following notation:

$E_\infty(D)$ denotes the Banach space of holomorphic functions over the disk $D=\{z\in\mathbb{C}:|z-1|<3/2\}$, and $E_{1,\infty}(D)$ is the Banach space of holomorphic functions over the disk D, which together with their first derivatives are continuous on $\bar{D}$, together with the sup norm $\|f\|=\max\{\sup_{z\in\bar{D}}|f(z)|, \sup_{z\in\bar{D}}|f'(z)|\}$. Therefore

Definition 3.1. We call the operator $\mathscr{L}_\beta^{(s)}$ on $E_\infty(D)$ defined by

$$\mathscr{L}_\beta^{(s)} f(z)=\sum_{i=1}^{\infty}(-1)^s\left[\frac{1}{z+i}\right]^{2\beta+2s} f\left(\frac{1}{z+i}\right) \quad s\in\{0,1\} \tag{3.4}$$

the generalized Perron–Frobenius (P-F in short) operator.

In our context we need to investigate the operator $\mathscr{L}_\beta^{(0)}$. This operator becomes a nuclear operator when restricted to the Banach space $E_\infty(D)$. The transfer operator $\mathscr{L}_\beta^{(0)}$ is a bounded linear operator on this space and it is straightforward to extend the above properties of $\mathscr{L}_\beta^{(0)}$ from $E_\infty(D)$ to the space $E_{1,\infty}(D)$. In the fact $\mathscr{L}_\beta^{(0)}$ can be decomposed as $\mathscr{L}_\beta^{(0)}=P_\beta+\mathscr{N}_\beta$, where P_β is the projector onto the eigenfunction $h_\beta(z)=1/(z+1)\log 2$ corresponding to the eigenvalue $\lambda_1(\beta)$. Its explicit form is given as $P_\beta f(z)=h_\beta(z)\int_0^1 f(x)\,dx$. $\mathscr{N}_\beta$ is some bounded linear operator with $P_\beta\mathscr{N}_\beta=\mathscr{L}_\beta^{(0)}=0$ and having spectral radius strictly smaller than $\lambda_1(\beta)$.

From the spectral properties of the operator $\mathscr{L}_\beta^{(0)}$ in the space $E_\infty(D)$, we have:

Lemma 3.2. The operator $\mathscr{L}_\beta^{(0)}: E_\infty(D)\to E_\infty(D)$ has a positive leading eigenvalue $\lambda_1(\beta)$, which is simple and strictly larger than all other eigenvalues in absolute value.

For the reader's convenience, we give a proof of Lemma 3.2 in the Appendix. The proof is based on positivity properties of the operator $\mathscr{L}_\beta^{(0)}$.

Before proving our main result, let us recall Kuzmin's Theorem, which is the cornerstone of the proof of Theorem 2.1. We refer the reader to ref. 11 for the proof.

Corollary 3.3. If $\mathscr{L}_\beta^{(0)}$ is the P-F operator for the Gauss map in the space $E_\infty(D)$, then

$$\|\lambda_1(\beta)^{-n}\,\mathscr{L}_\beta^{(0)\,n}-\mathscr{P}_\beta\|\leqslant q_\beta^n \tag{3.5}$$

where $q_\beta^n=|\lambda_2(\beta)/\lambda_1(\beta)|<1$, $\lambda_2(\beta)$ the second highest eigenvalue of $\mathscr{L}_\beta^{(0)}$ in absolute value, and $\|\cdot\|$ the norm associated with the canonical scalar product on $E_\infty(D)$.

4. PROOF OF THE MAIN RESULT

The proof of Theorem 2.1 will be done in four steps as mentioned before.

Step 1. Decorrelation Property. As already mentioned in the introduction, to prove the approximation by diffusion we shall use the property analogous to the "very weak-Bernoulli" property. First we refer to the above section to reformulate the decorrelation property (2.14). Now

$$\left\langle \nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2} f(Y_s)\,ds\right):\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\,ds\right)^{\otimes 2}\right\rangle$$

$$=\int_{\tilde{\mathcal{O}}}\left[\nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2} f\circ\chi^{-1}(S_s(z,\sigma))\,ds\right):\right.$$

$$\left.\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f\circ\chi^{-1}(S_s(z,\sigma))\,ds\right)^{\otimes 2}\right]d\tilde{\nu}(z,\sigma) \tag{4.1}$$

where the map χ is defined in Paragraph 3.2.

Set

$$R(x)=\nabla_x^2\phi(x+\cdot),\qquad L(z)=z^{\otimes 2} \tag{4.2}$$

We then easily see that Eq. (4.1) can be transformed into

$$\int_{\tilde{\mathcal{O}}}\left[\nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2} f\circ\chi^{-1}(S_s(z,\sigma))\,ds\right):\right.$$

$$\left.\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f\circ\chi^{-1}(S_s(z,\sigma))\,ds\right)^{\otimes 2}\right]d\tilde{\nu}(z,\sigma)$$

$$=\int_{\mathcal{O}}\left[\int_0^{g(z)} R\left(\varepsilon\int_0^{t/\varepsilon^2}\tilde{f}(S_s(z,\sigma))\,ds\right)\right.$$

$$\left.\times L\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2}\tilde{f}(S_s(z,\sigma))\,ds\right)d\sigma\right]d\nu(z) \tag{4.3}$$

Next, decompose the interval $[0,t]$ in the form:

$$[0,t]=[0,g(z)-\sigma]\cup[g(z)-\sigma,\ g(Tz)+g(z)-\sigma]$$

$$\cup[g(Tz)+g(z)-\sigma,\ g(T^2z)+g(Tz)+g(z)-\sigma]$$

$$\cup\cdots\cup[-\sigma+g(z)\cdots g(T^{k-1}z),\ t] \tag{4.4}$$

Under this decomposition, the integral $\int_0^t \tilde{f}(S_s(z,\sigma))\,ds$ in expansion of the transformation T reads as:

$$\begin{aligned}
&\int_0^t \tilde{f}(S_s(z,\sigma))\,ds\\
&\quad = \int_0^{g(z)-\sigma} \tilde{f}(z,\sigma+s)\,ds + \int_{g(z)-\sigma}^{g(Tz)+g(z)-\sigma} \tilde{f}(Tz,\sigma+s-g(z))\,ds\\
&\qquad + \int_{g(Tz)+g(z)-\sigma}^{g(T^2z)+g(Tz)+g(z)-\sigma} \tilde{f}(T^2z,\sigma+s-g(z)-g(Tz))\,ds\\
&\qquad + \cdots + \int_{g(T^{k-1}z)+\cdots+g(z)-\sigma}^{g(T^kz)+\cdots+g(Tz)+g(z)-\sigma} \tilde{f}(T^kz,\sigma+s-g(z)-g(Tz)\\
&\qquad - \cdots - g(T^{k-1}z))\,ds
\end{aligned}$$

in the above decomposition, it is to be recalled that $t = -\sigma + g(Tz) + \cdots + g(T^{k-1}z) + g(T^kz)$. A change of variable yields

$$\int_0^t \tilde{f}(S_s(z,\sigma))\,ds = \sum_{-\sigma+g(Tz)+\cdots+g(T^{k-1}z)+g(T^kz)\leqslant t} \int_0^{g(T^kz)} \tilde{f}(T^kz,s')\,ds' \tag{4.5}$$

Therefore, set

$$F(z') = \int_0^{g(z')} \tilde{f}(z',s')\,ds'$$

to obtain

$$\int_0^t \tilde{f}(S_\sigma(z,s))\,ds = \sum_{-\sigma+g(z)+\cdots+g(T^{k-1}z)+g(T^kz)\leqslant t} F(T^kz')$$

The last expression induces the investigation of the terms of the form:

$$\sum_{g(z)+\cdots+g(T^{k-1}z)+g(T^kz)\leqslant t+s} F(T^kz) \tag{4.6}$$

In order to study these terms, we reformulate Eq. (4.3) in the following form:

$$\int_{\mathcal{O}}\left[\int_0^{g(z)} R\left(\varepsilon\int_0^{t/\varepsilon^2}\tilde{f}(S_\sigma(z,s))\,ds\right) L\left(\varepsilon\int_{(t+\delta)/\varepsilon^2}^{(t+\tau)/\varepsilon^2}\tilde{f}(S_\sigma(z,s))\,ds\right)d\sigma\right]d\nu(z)$$
$$=\int_{\mathcal{O}}\left[\int_0^{g(z)} R\left(\varepsilon\sum_{g(z)+\cdots+g(T^{k-1}z)+g(T^kz)\leqslant t/\varepsilon^2+s} F(T^kz)\right):\right.$$
$$\left. L\left(\varepsilon\sum_{g(T^{k+l}z)+\cdots+g(T^{k+l+m}z)\leqslant (t+\tau)/\varepsilon^2+s} F(T^{l+k}z)\right)ds\right]d\nu(z) \tag{4.7}$$

With the above transformation, we can state the following proposition which contains the key to the decorrelation property:

Proposition 4.1. Let S_t be the special flow defined by Eq. (3.2), and two functions $f, g: \mathcal{M}\to\mathbb{R}^n$. Let R and L be defined by Eq. (4.2). Then

$$\int_{\tilde{\mathcal{O}}}\left[R\left(\varepsilon\int_0^{t/\varepsilon^2} f(S_\sigma(z,s))\,d\sigma\right): L\left(\varepsilon\int_{t_1/\varepsilon^2}^{t_2/\varepsilon^2} g(S_\sigma(z,s))\,d\sigma\right)\right]d\tilde{\nu}(z,s)$$
$$-\int_{\tilde{\mathcal{O}}} R\left(\varepsilon\int_0^{t/\varepsilon^2} f(S_\sigma(z,s))\,d\sigma\right)d\tilde{\nu}(z,s):$$
$$\int_{\tilde{\mathcal{O}}} L\left(\varepsilon\int_{t_1/\varepsilon^2}^{t_2/\varepsilon^2} g(S_\sigma(z,s))\,d\sigma\right)d\tilde{\nu}(z,s)\to 0 \tag{4.8}$$

as $\varepsilon\to 0$, for all $0<t<t_1<t_2$.

Proof. The proof of this proposition uses exactly the same ideas introduced by Katznelson in ref. 10, but here we need a more precise results, which will be proven below, and based on the decomposition of the intervals $[0, t_1/\varepsilon^2]$, $[t_1/\varepsilon^2, t_2/\varepsilon^2]$ as in Eq. (4.4). Using Eq. (4.7) we have reduced the problem to showing that

$$\int_{\mathcal{O}}\left[\int_0^{g(z)} R\left(\varepsilon\sum_{g(z)+\cdots+g(T^{k-1}z)+g(T^kz)\leqslant t/\varepsilon^2+s} F(T^kz)\right):\right.$$
$$\left. L\left(\varepsilon\sum_{s+t_1/\varepsilon^2<g(T^{k+l}z)+\cdots+g(T^{k+l+m}z)<t_2/\varepsilon^2+s} F(T^{k+l}z)\right)ds\right]d\nu(z,s)$$
$$-\int_{\mathcal{O}}\int_0^{g(z)} R\left(\varepsilon\sum_{g(z)+\cdots+g(T^{k-1}z)+g(T^kz)\leqslant t/\varepsilon^2+s} F(T^kz)\right)ds\,d\nu(z,s):$$
$$\int_{\mathcal{O}} L\left(\varepsilon\sum_{s+t_1/\varepsilon^2<g(T^{k+l}z)+\cdots+g(T^{k+l+m}z)\leqslant t_2/\varepsilon^2+s} F(T^{k+l}z)\right)d\nu(z,s)\to 0 \tag{4.9}$$

as $\varepsilon\to 0$.

Moreover, the right hand side of Eq. (4.9) can be expressed as

$$\int_{\mathcal{O}} \int_0^{g(z)} A_s(z, Tz,..., T^k z)\, B_s(T^{k+l}z,..., T^{k+l+m}z)\, d\sigma\, d\nu(z)$$

$$-\int_{\mathcal{O}} \int_0^{g(z)} A_s(z, Tz,..., T^k z)\, d\sigma\, d\nu(z)$$

$$\times \int_{\mathcal{O}} B_s(T^{k+l}z,..., T^{k+l+m}z)\, d\nu(z) = 0 \tag{4.10}$$

where we denote A_s and B_s by

$$A_s(z, Tz,..., T^k z) = R\left(\varepsilon \sum_{g(z)+\cdots+g(T^{k-1}z)+g(T^k z) \leqslant t/\varepsilon^2+s} F(T^k z)\right)$$

and

$$B_s(T^{k+l}z,..., T^{k+l+m}z)$$

$$= L\left(\varepsilon \sum_{s+t_1/\varepsilon^2 < g(T^{k+l}z)+\cdots+g(T^{k+l+m}z) \leqslant t_2/\varepsilon^2+s} F(T^{k+l}z)\right) \tag{4.11}$$

It is convenient now to introduce the following notation. We denote by $|\cdot|_\infty$ the supremum norm with respect to the canonical basis of $\mathbb{R}^n$. If $f: (\Omega, \mathscr{F}) \to (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ is a measurable map, we define

$$\|f\|_\infty = \|f\|_{L^\infty} := \inf\{M > 0 : \mu\{x\ |f(x)|_\infty > M\} = 0\}$$

and

$$1 \leqslant p < \infty, \qquad \|f\|_p = \|f\|_{L^p(\mu)} = |\langle |f|^p \rangle_\infty^{1/p}$$

We begin by establishing the following result:

Lemma 4.2. For s small enough, the quantities A_s and B_s defined by Eqs. (4.11) are independent of s.

Proof. In order to prove this lemma, we shall prove that the error

$$\sum_{g(z)+\cdots+g(T^k z) \leqslant t/\varepsilon^2+s} F(T^k z) - \sum_{g(z)+\cdots+g(T^k z) \leqslant t/\varepsilon^2} F(T^k z)$$

is dependent on epsilon. Multiplying the above expansion by ε and rewriting it in terms of Eq. (4.5), we get

$$\varepsilon \cdot \left(\sum_{g(z)+\cdots+g(T^k z) \leqslant t/\varepsilon^2 + s} F(T^k z) - \sum_{g(z)+\cdots+g(T^k z) \leqslant t/\varepsilon^2} F(T^k z) \right) = \varepsilon \int_0^{t/\varepsilon^2} \left[f(S_\sigma(z,s)) - f(S_\sigma(z,0)) \right] d\sigma \tag{4.12}$$

The right side of Eq. (4.12) then becomes

$$\varepsilon \int_0^{t/\varepsilon^2} \left[f(S_\sigma(z,s)) - f(S_\sigma(z,0)) \right] d\sigma = \varepsilon \int_0^{t/\varepsilon^2} \left[f(S_{\sigma+s}(z,0)) - f(S_\sigma(z,0)) \right] d\sigma, \qquad 0 \leqslant s < g(z) \tag{4.13}$$

A change of variable leads to

$$\varepsilon \int_0^{t/\varepsilon^2} f(S_\sigma(z,s))\, d\sigma = \varepsilon \int_s^{t/\varepsilon^2 + s} f(S_\sigma(z,0))\, d\sigma$$

Therefore

$$\varepsilon \int_0^{t/\varepsilon^2} f(S_\sigma(z,s))\, d\sigma - \varepsilon \int_0^{t/\varepsilon^2} f(S_\sigma(z,0))\, d\sigma = \varepsilon \int_{t/\varepsilon^2}^{t/\varepsilon^2 + s} f(S_\sigma(z,s))\, d\sigma - \varepsilon \int_0^{s} f(S_\sigma(z,0))\, d\sigma \tag{4.15}$$

Taking the norm of Eq. (4.15) yields

$$\left| \varepsilon \int_{t/\varepsilon^2}^{t/\varepsilon^2 + s} f(S_\sigma(z,s))\, d\sigma - \varepsilon \int_0^{s} f(S_\sigma(z,0))\, d\sigma \right| \leqslant 2\varepsilon \cdot s \cdot \|f\|_{L^\infty} \tag{4.16}$$

with $s < g(z)$ for s small enough.

Define

$$\mathcal{N}^{\varepsilon}_{t,z,s} = \{ k \mid g(z) + \cdots + g(T^k z) \leqslant t/\varepsilon^2 + s \} = [0, k^{\varepsilon}_{\mathrm{Max}}(t,z,s)] \cap \mathbb{N} \tag{4.17}$$

and examinate the expansion (4.12). For $k = k^{\varepsilon}_{\mathrm{Max}}(t,z,s) - k^{\varepsilon}_{\mathrm{Max}}(t,z,0)$, we get after some calculations,

$$\sum_{g(z)+\cdots+g(T^k z)\leqslant t/\varepsilon^2+s} F(T^k z) - \sum_{g(z)+\cdots+g(T^k z)\leqslant t/\varepsilon^2} F(T^k z)$$
$$= \sum_{k=k^\varepsilon_{\text{Max}}(t,z,0)+1}^{k^\varepsilon_{\text{Max}}(t,z,s)} F(T^k z)$$

where $F(z')$ is defined by Eq. (4.5). Thus we must investigate the following three cases:

1st Case. Assume that $z' = T^k z$ is far away from 0. As $g(z) := \ln z$, there exists a constant C such that,

$$\{m \mid g(z)+\cdots+g(T^m z) < t/\varepsilon^2\} \sim Cm$$

holds for m large enough.

Hence, $g(z') = g(T^k z)$ exists and $\int_0^{g(T^k z)} \tilde{f}(T^k z, s')\,ds' < \infty$. In other words,

$$\sum_{g(z)+\cdots+g(T^k z)<t/\varepsilon^2+s} F(T^k z) - \sum_{g(z)+\cdots+g(T^k z)<t/\varepsilon^2} F(T^k z) = O(1) \tag{4.18}$$

2nd Case. Assume next that z' is in the neighbourhood 0, i.e., the case where $g(0) = -\infty$. In this case, if $T^k z \sim 0$, then $k \sim k_{\text{Max}}$ and $T^{k^\varepsilon_{\text{Max}}(t,z,s)} z \sim 0$.

3nd Case. Finally, if $k^\varepsilon_{\text{Max}}(t,z,s) < z' < k^\varepsilon_{\text{Max}}(t,z,0)$, then $g(T^k z) \sim 0$.

In all cases, we get

$$\sum_{g(z)+\cdots+g(T^k z)<t/\varepsilon^2+s} F(T^k z) - \sum_{g(z)+\cdots+g(T^k z)<t/\varepsilon^2} F(T^k z) = O(1) \tag{4.19}$$

Next, assume that R is a Lipschitz function having the Lipschitz constant K_R. Obviously

$$\left| \int_{\tilde{\mathcal{O}}} R\left(\varepsilon \int_0^{t/\varepsilon^2} f(S_\sigma(z,s))\,d\sigma\right) d\tilde{v}(z) - \int_{\tilde{\mathcal{O}}} d\tilde{v}(z)\, R\left(\varepsilon \int_0^{t/\varepsilon^2} f(S_\sigma(z,0))\,d\sigma\right)\right|$$
$$\leqslant \int_{\tilde{\mathcal{O}}} d\tilde{v}(z)\, K_R \cdot 2\varepsilon\, \|f\|_{L^\infty} \cdot s \leqslant K_R\, \|f\|_{L^\infty} \cdot \varepsilon \int_{\mathcal{O}} dv(z) \int_0^{g(z)} s\,ds$$
$$\leqslant K_R\, \|f\|_{L^\infty} \cdot \varepsilon \int_{\mathcal{O}} g^2(z)\,dv(z) \leqslant \varepsilon C K_R\, \|f\|_{L^\infty} \tag{4.20}$$

which means that A_s and B_s are independent of s. Integrating Eq. (4.10) with respect to s, we reduce

$$\int_{\mathcal{O}} g(z)\, A(z, Tz,..., T^k z)\, B(T^{k+l}z,..., T^{k+l+m}z)\, dv(z)$$
$$-\int_{\mathcal{O}} g(z)\, A(z, Tz,..., T^k z)\, dv(z) \int_{\mathcal{O}} B(T^{k+l}z,..., T^{k+l+m}z)\, dv(z) = 0 \tag{4.21}$$

The proof of Lemma 4.2. is now complete. ▮

End of the Proof of Proposition 4.1. We are in a position to conclude the proof of Proposition 4.1. Introduce a partition $\mathcal{P}$ such that

$$\mathcal{P}_0^k = \mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-k}\mathcal{P} \qquad \text{and}$$
$$\mathcal{P}_{k+l}^{k+l+m} = T^{-k-l}\mathcal{P} \vee \cdots \vee T^{-k-l-m}\mathcal{P}$$

and

$$\text{(H3)} \quad \sum_{E \in \mathcal{P}} \inf_E |\ln v(E)| = \sum_{E \in \mathcal{P}} \frac{1}{v(E)} \int_E \ln(v(E)) \sim \frac{1}{\ln 2} \int_0^1 \frac{\ln z}{1+z}\, dz$$

Let E and F be the finite decompositions of $[0, 1[$ into elements of $\mathcal{P}_0^k$ and $\mathcal{P}_{k+l}^{k+l+m}$. Write A and B in the form

$$\begin{aligned} A(z, Tz,..., T^k z) &= \sum_{E \in \mathcal{P}_0^k} A_{|E} \mathbf{1}_E \\ B(T^{k+l}z,..., T^{k+l+m}z) &= \sum_{F \in \mathcal{P}_{k+l}^{k+l+m}} B_{|F} \mathbf{1}_F \end{aligned} \tag{4.22}$$

First observe that, since μ is ϕ-invariant

$$\begin{aligned} \mu(E \cap \phi^{-n}F) &= \mu(\mathbf{1}_E \cdot \mathbf{1}_{\phi^{-n}F}) \\ &= \mu(\mathbf{1}_E \cdot (\mathbf{1}_F \circ \phi^n)) \\ &= v(h_\beta \mathbf{1}_E \cdot (\mathbf{1}_F \circ \phi^n)) \\ &= \lambda_1(\beta)^{-n}\, \mathcal{L}_\beta^{*n} v(h_\beta \mathbf{1}_E \cdot (\mathbf{1}_F \circ \phi^n)) \\ &= v(\lambda_1(\beta)^{-n}\, \mathcal{L}_\beta^n (h_\beta \mathbf{1}_E \cdot (\mathbf{1}_F \circ \phi^n))) \\ &= v(\lambda_1(\beta)^{-n}\, \mathcal{L}_\beta^n (h_\beta \mathbf{1}_E) \cdot \mathbf{1}_F) \end{aligned}$$

Using this fact,

$$\begin{aligned}|\mu(E \cap \phi^{-n}F) - \mu(E)\,\mu(F)| &= |\mu(E \cap \phi^{-n}F) - \nu(h_\beta \mathbf{1}_E)\,\nu(h_\beta \mathbf{1}_F)| \\ &= |\nu(\lambda_1(\beta)^{-n}\,\mathscr{L}_\beta^{(0)\,n}(h_\beta \mathbf{1}_E \cdot \mathbf{1}_F)) - \nu(h_\beta \mathbf{1}_E)\,\nu(h_\beta \mathbf{1}_F)| \\ &= |\nu((\lambda_1(\beta)^{-n}\,\mathscr{L}_\beta^{(0)\,n}(h_\beta \mathbf{1}_E) - \nu(h_\beta \mathbf{1}_E)\,h_\beta)\,\mathbf{1}_F)| \\ &\leqslant \|\lambda_1(\beta)^{-n}\,\mathscr{L}_\beta^{(0)\,n}(h_\beta \mathbf{1}_E) - \nu(h_\beta \mathbf{1}_E)\,h_\beta\|\;\nu(F)\end{aligned}$$

Hence, taking into account Corollary 3.3

$$\|\lambda_1(\beta)^{-n}\,\mathscr{L}_\beta^{(0)\,n}(h_\beta \mathbf{1}_E) - \nu(h_\beta \mathbf{1}_E)\,h_\beta\| \leqslant C\mu(E)\,\rho^{n-s}$$

where C is a constant and $\rho \in \,]0, 1[$ is defined as in Lemma 3.2. Then

$$|\mu(E \cap \phi^{-n}F) - \mu(E)\,\mu(F)| \leqslant C(\inf h_\beta)^{-1}\,\mu(E)\,\mu(F)\,\rho^{n-s} \tag{4.23}$$

for $n \geqslant s$. Thus $\mu(E \cap \phi^{-n}F) \to \mu(E)\,\mu(F)$.

Now we examine the decorrelation property in terms of tensor-products[3] with the aid of (H3). Let f_i be regular functions. Then Eq. (4.21) can be replaced by

$$\begin{aligned}\Delta(f_i) = &\int f_0(z) \cdots f_k(T^k z)\, f_{k+l}(T^{k+l} z) \cdots f_{k+l+m}(T^{k+l+m} z)\, d\nu(z) \\ &- \int f_0(z) \cdots f_k(T^k z)\, d\nu(z) \\ &\times \int f_{k+l}(T^{k+l} z) \cdots f_{k+l+m}(T^{k+l+m} z)\, d\nu(z)\end{aligned} \tag{4.24}$$

Define

$$Osc(f_{i\,|\,E}) = \sup_{z,\, z' \in E} |f_i(z) - f_i(z')|$$

and let $\mathscr{P}$ be the partition (H3) such that, for $\eta > 0$,

$$\sup_{\substack{i \\ E \in \mathscr{P}}} Osc(f_{i|E}) < \eta \tag{4.25}$$

[3] Let Ω be an open set of $\mathbb{R}^n$, Ω' an open set of $\mathbb{R}^p$, f a function defined on Ω, g defined on Ω', x the variable in $\mathbb{R}^n$ and y the variable in $\mathbb{R}^p$. We call tensor-product of f and g the function defined on $\Omega \times \Omega'$ by $f \otimes g(x, y) \stackrel{\text{def}}{=} f(x)\, g(y)$.

Replacing f_i by

$$\tilde{f}_i = \sum_{E \in \mathscr{P}} \frac{1}{\nu(E)} \int_E f_i \, d\nu \cdot \mathbf{1_E} \tag{4.26}$$

we will estimate

$$|\Delta(f_i) - \Delta(\tilde{f}_i)| + |\Delta(\tilde{f}_i)| \tag{4.27}$$

Firstly, using Eq. (4.23),

$$\begin{aligned}
|\Delta(\tilde{f}_i)| &= \bigg| \int \tilde{f}_0(z) \cdots \tilde{f}_k(T^k z)\, \tilde{f}_{k+l}(T^{k+l} z) \cdots \tilde{f}_{k+l+m}(T^{k+l+m} z)\, d\nu(z) \\
&\quad - \int \tilde{f}_0(z) \cdots \tilde{f}_k(T^k z)\, d\nu(z) \\
&\quad \times \int \tilde{f}_{k+l}(T^{k+l} z) \cdots \tilde{f}_{k+l+m}(T^{k+l+m} z)\, d\nu(z) \bigg| \\
&\leqslant C\rho^l \sum_{\substack{E \in \mathscr{P}_0^k \\ F \in \mathscr{P}_{k+l}^{k+l+m}}} \prod_{i=0}^{k} \tilde{f}_{i|_E} \prod_{i=k+l}^{k+l+m} \tilde{f}_{i|_F}\, \nu(E)\, \nu(F) \\
&= C\rho^l \int \tilde{f}_0(z) \cdots \tilde{f}_{k+l}(T^{k+l} z) \cdots \tilde{f}_{k+l+m}(T^{k+l+m} z)\, d\nu(z)
\end{aligned} \tag{4.28}$$

where $\rho \equiv q_\beta$ is defined as in Corollary 3.3 and the $\tilde{f}_i$ are measurable functions. Equation (4.28) allows us to have L^1-control on $\tilde{f}_i$ instead L^∞. On the other hand, it is not hard to see that

$$\begin{aligned}
|\Delta(f_i) - \Delta(\tilde{f}_i)| &\leqslant \bigg| \int f_0(z) \cdots f_k(T^k z)\, f_{k+l}(T^{k+l} z) \cdots f_{k+l+m}(T^{k+l+m} z)\, d\nu(z) \\
&\quad - \int \tilde{f}_0(z) \cdots \tilde{f}_{k+l}(T^{k+l} z) \cdots \tilde{f}_{k+l+m}(T^{k+l+m} z)\, d\nu(z) \bigg| \\
&\quad + \bigg| \int f_0(z) \cdots f_k(T^k z)\, d\nu(z) \\
&\quad \times \int f_{k+l}(T^{k+l} z) \cdots f_{k+l+m}(T^{k+l+m} z)\, d\nu(z) \\
&\quad - \int \tilde{f}_0(z) \cdots \tilde{f}_k(T^k z)\, d\nu(z) \\
&\quad \times \int \tilde{f}_{k+l}(T^{k+l} z) \cdots \tilde{f}_{k+l+m}(T^{k+l+m} z)\, d\nu(z) \bigg| \\
&= I_1 + I_2
\end{aligned} \tag{4.29}$$

Note that

$$I_1 \leqslant \left| \int (f_0 - \tilde{f}_0)\, f_1\, dv(z) + \int (f_1 - \tilde{f}_1)\, f_2\, dv(z) + \cdots \right| \\ + \left| \int \tilde{f}_0 (f_1 - \tilde{f}_1)\, dv(z) \right| + \left| \int \tilde{f}_1 (f_2 - \tilde{f}_2)\, dv(z) \right| + \cdots \tag{4.30}$$

From this, since

$$\left| \int (f_0 - \tilde{f}_0)\, dv(z) \right| \leqslant \| f_0 - \tilde{f}_0 \|_{L^1} \tag{4.31}$$

we get

$$\left| \int \tilde{f}_0 (f_1 - \tilde{f}_1)\, dv(z) \right| + \left| \int \tilde{f}_0 (f_1 - \tilde{f}_1)\, dv(z) \right| + \left| \int \tilde{f}_1 (f_2 - \tilde{f}_2)\, dv(z) \right| + \cdots \\ \leqslant \| \tilde{f}_0 \|_{L^1}\, \| f_1 - \tilde{f}_1 \|_{L^\infty} + \cdots \tag{4.32}$$

Hence

$$\left| \int f_0(z) \cdots f_k(T^k z)\, f_{k+l}(T^{k+l} z) \cdots f_{k+l+m}(T^{k+l+m} z)\, dv(z) \right. \\ \left. - \int \tilde{f}_0(z) \cdots \tilde{f}_{k+l}(T^{k+l} z) \cdots \tilde{f}_{k+l+m}(T^{k+l+m} z)\, dv(z) \right| \\ \leqslant C_1\, \| f_0 - \tilde{f}_0 \|_{L^1} + C_2\, \| \tilde{f}_0 \|_{L^1}\, \| f_1 - \tilde{f}_1 \|_{L^\infty} \cdots \| f_{k+l+m} - \tilde{f}_{k+l+m} \|_{L^\infty} \tag{4.33}$$

Naturally, we have the estimate on the term I_2 by the same ansatz. Now, using Corollary 3.3 yields

$$\left| \int_{\mathcal{O}} g(z)\, A(z, Tz,..., T^k z B(T^{k+l} z,..., T^{k+l+m} z)\, dv(z) \right. \\ \left. - \int_{\mathcal{O}} g(z)\, A(z, Tz,..., T^k z)\, dv(z) \int_{\mathcal{O}} B(T^{k+l} z,..., T^{k+l+m} z)\, dv(z) \right| \\ \leqslant C_0\, \rho^l \int \tilde{f}_0(z) \cdots \tilde{f}_{k+l}(T^{k+l} z) \cdots \tilde{f}_{k+l+m}(T^{k+l+m} z)\, dv(z) \cdot v(E)\, v(F) \\ + C_1\, \| f_0 - \tilde{f}_0 \|_{L^1} + C_2\, \| f_0 - \tilde{f}_0 \|_{L^1}\, \| f_1 - \tilde{f}_1 \|_{L^\infty} \cdots \| f_{k+l+m} - \tilde{f}_{k+l+m} \|_{L^\infty} \tag{4.34}$$

where $C_0, C_1,\ldots$ denote some constants. To complete, the proof summing over $E \in \mathscr{P}_0^k$ and $F \in \mathscr{P}_{k+l}^{k+l+m}$, inserting this result and Eq. (4.28) into Eq. (4.27) yields

$$\begin{aligned}
&\bigg| \int_{\mathcal{O}} \int_0^{g(z)} A_s(z, Tz,\ldots, T^k z)\, B_s(T^{k+l}z,\ldots, T^{k+l+m}z)\, d\sigma\, dv(z) \\
&\quad - \int_{\mathcal{O}} \int_0^{g(z)} A_s(z, Tz,\ldots, T^k z)\, dv(z) \int_{\mathcal{O}} B_s(T^{k+l}z,\ldots, T^{k+l+m}z)\, ds\, dv(z) \bigg| \\
&\leqslant C_0 \rho^l \|\tilde{f}_0\|_{L^1} \|\tilde{f}_1\|_{L^\infty} \cdots \|\tilde{f}_{k+l+m}\|_{L^\infty} + C_1 \|f_0 - \tilde{f}_0\|_{L^1} \\
&\quad + C_2 \|f_0 - \tilde{f}_0\|_{L^1} \|f_1 - \tilde{f}_1\|_{L^\infty} \cdots \|f_{k+l+m} - \tilde{f}_{k+l+m}\|_{L^\infty}
\end{aligned} \tag{4.35}$$

To summarize, we have used Eq. (4.25) which interprets the smallness of the oscillation (4.25) and we replace the suspension $g(z)$ by f_0 which we have assumed only integrable. The proof of Proposition 4.1. is now complete. ∎

Decorrelation in Eq. (2.13). Observe that since the treatment of the linear term is simpler (but follows the same lines) than the treatment of the quadratic term, only this last one, will be considered in details. The key idea to prove decorrelation in this term is to introduce a positive "small" time δ, or alternatively some integer k' in Eq. (4.10). We start with

$$\begin{aligned}
&\left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s)\, ds \right) : \left(\varepsilon \int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\, ds \right)^{\otimes 2} \right\rangle \\
&\quad - \left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s)\, ds \right) : \left(\varepsilon \int_{(t+\delta)/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\, ds \right)^{\otimes 2} \right\rangle \\
&= \left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s)\, ds \right) : \left(2\varepsilon \int_{t/\varepsilon^2}^{(t+\delta)/\varepsilon^2} f(Y_s)\, ds \vee \varepsilon \int_{(t+\delta)/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\, ds \right) \right. \\
&\quad \left. + \left(\varepsilon \int_{t/\varepsilon^2}^{(t+\delta)/\varepsilon^2} f(Y_s)\, ds \right)^{\otimes 2} \right\rangle
\end{aligned} \tag{4.36}$$

where we denote

$$a \vee b = \tfrac{1}{2}(a \otimes b + b \otimes a)$$

Now, observe that

$$\left\| \varepsilon \int_{t/\varepsilon^2}^{(t+\delta)/\varepsilon^2} f(Y_s)\, ds \right\|_{L^2} = \left\| \varepsilon \int_0^{\delta/\varepsilon^2} f(Y_s)\, ds \right\|_{L^2} = O(\sqrt{\delta}) \tag{4.37}$$

Using Cauchy–Schwartz's inequality in Eq. (4.36), we obtain

$$
\begin{aligned}
&\left|\left\langle \nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2} f(Y_s)\,ds\right) : \left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\,ds\right)^{\otimes 2}\right\rangle\right.\\
&\qquad\left.-\left\langle \nabla_x^2\phi\left(x+\varepsilon\int_0^{t/\varepsilon^2} f(Y_s)\,ds\right) : \left(\varepsilon\int_{(t+\delta)/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\,ds\right)^{\otimes 2}\right\rangle\right|\\
&\quad\leqslant \left\langle \left|\nabla_x^2\phi\left(x+\int_0^{t/\varepsilon^2} f(Y_s)\,ds\right)\right| \left|\varepsilon\int_{t/\varepsilon^2}^{(t+\delta)/\varepsilon^2} f(Y_s)\,ds\right|^2\right\rangle\\
&\qquad+2\left\langle\left|\nabla_x^2\phi\left(x+\int_0^{t/\varepsilon^2} f(Y_s)\,ds\right)\right|\right.\\
&\qquad\left.\times\left|\varepsilon\int_{t/\varepsilon^2}^{(t+\delta)/\varepsilon^2} f(Y_s)\,ds\right|^2 \left|\varepsilon\int_{(t+\delta)/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\,ds\right|^2\right\rangle
\end{aligned}
\tag{4.38}
$$

The first term of Eq. (4.38), with $T=t/\varepsilon^2$ is bounded by

$$
\begin{aligned}
&\|\nabla_x^2\phi\|_\infty \left|\varepsilon\int_{t/\varepsilon^2}^{(t+\delta)/\varepsilon^2} f(Y_s)\,ds\right|_2^2\\
&\qquad\leqslant \|\nabla_x^2\phi\|_\infty \sup_T \left\|\frac{1}{\sqrt{T}}\int_0^T f(Y_s)\,ds\right\|_4^2 (2\sqrt{2\delta}+\delta+\varepsilon^2)
\end{aligned}
$$

and the second term is bounded by

$$
\|\nabla_x^2\phi\|_\infty \sup_T \left\|\frac{1}{\sqrt{T}}\int_0^T f(Y_s)\,ds\right\|_4^2 (2\sqrt{2\delta}+\delta+2\varepsilon^2)
$$

Step 2. Estimate of the remainder. We have

$$
\begin{aligned}
r_{\varepsilon,\tau,t} &:= \left\langle\left|\left(\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\,ds\right)\right|^3\right\rangle\\
&\leqslant \left\langle\left|\left(\varepsilon\int_0^{(t+\tau)/\varepsilon^2} f(Y_s)\,ds\right)\right|^3\right\rangle + O(\varepsilon^3)
\end{aligned}
\tag{4.39}
$$

or using Hölder's inequality with $T=\tau/\varepsilon^2$ leads to

$$
r_{\varepsilon,\tau,t} \leqslant C(\varepsilon\sqrt{T})^3 \left(\int_{\mathcal{O}} \left|\frac{1}{\sqrt{T}}\int_0^T f(Y_s)\,ds\right|^4 dv(z)\right)^{3/4} + O(\varepsilon^3) \tag{4.40}
$$

Hence, letting $\varepsilon \to 0$ yields

$$\limsup_{\varepsilon \to 0} \sup_{t \in \mathbb{R}^+} \left\langle \left| \left(\varepsilon \int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\, ds \right) \right|^3 \right\rangle = O(\tau)^{1/2} \tag{4.41}$$

Step 3. We begin with

$$\left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s)\, ds \right) : \left(\varepsilon \int_{(t+\delta)/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\, ds \right)^{\otimes 2} \right\rangle$$

$$= \int_{\mathcal{O}} dv(z) \int_0^{g(z)} A_s(z, Tz,..., T^k)\, B_s(T^{k'+l}z,..., T^{k+l+m}z)\, ds$$

$$= \int_{\mathcal{O}} g(z)\, A(z, Tz,..., T^k)\, dv(z)\, B(T^{k'+l}z,..., T^{k+l+m}z)\, dv(z) \tag{4.42}$$

Observe that

$$\int_{\mathcal{O}} g(z)\, A(z, Tz,..., T^k)\, dv(z) \simeq \int_{\mathcal{O}} \int_0^{g(z)} A_s(z, Tz,..., T^k)\, d\tilde{v}(z, s) \tag{4.43}$$

and similarly

$$\int_{\mathcal{O}} B_s(T^{k'+l}z,..., T^{k+l+m}z)\, dv(z) \simeq \int_{\mathcal{O}} B(z, Tz,..., T^m z)\, dv(z) \tag{4.44}$$

by the invariance of the transformation T. It now follows that

$$\left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s)\, ds \right) : \left(\varepsilon \int_{(t+\delta)/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\, ds \right)^{\otimes 2} \right\rangle$$

$$\simeq \int_{\mathcal{O}} \int_0^{g(z)} A(z, Tz,..., T^k)\, d\tilde{v}(z, s) \int_{\mathcal{O}} B(z, Tz,..., T^m z)\, dv(z) \tag{4.45}$$

$$= \tfrac{1}{2} \tau\, \kappa^2 \colon \nabla_x^2 \langle \varphi_\varepsilon(t, x) \rangle \tag{4.46}$$

Step 4. Taking the Limit. *(a) Uniform Compactness Result.* The following lemma will be important in what follows.

Lemma 4.3. Assume that ϕ is a function of class $C_b^3(\mathbb{R}^n)$ and $f\colon \mathcal{M} \to \mathbb{R}^n$ be an integrable function such that, $\sup_T \|(1/\sqrt{T}) \int_0^T f(Y_s)\, ds\|_{L^1} < +\infty$. Denote by u_ε the family of function $\langle \psi_\varepsilon(t, x, \cdot) \rangle$. Then

(a) $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $C_b^{0,3}(\mathbb{R}^+\times\mathbb{R}^n)$;

(b) there exists a constant $C>0$ such that, for all $\varepsilon>0$, $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ with $t+\tau\geqslant 0$, and $k=0,1,2$, and $x\in\mathbb{R}^n$, we have $|\nabla_x^k u_\varepsilon(t+\tau,x)-\nabla_x^k u_\varepsilon(t,x)|_\infty\leqslant C\sqrt{|\tau|}$;

(c) $(u_\varepsilon)_{\varepsilon>0}$ is relatively compact in $C_b^{0,2}([0,\tau]\times X)$, for $\tau>0$, and for any compact X in $\mathbb{R}^n$.

Proof. Proof of (a). For $k\in\mathbb{N}^n$ such that $|k|\leqslant 3$, we have $\sup_t\sup_x|D_x^k u_\varepsilon(t,x)|\leqslant\|D^k\phi\|_\infty$.

Proof of (b). Let $\tau>0$ and $k=0,1,2$. By the dominated convergence theorem, we have

$$
\begin{aligned}
|\nabla_x^k u_\varepsilon(t+\tau,x)-\nabla_x^k u_\varepsilon(t,x)|_\infty &\leqslant n\,\|\nabla^{k+1}\phi\|_\infty\left\|\varepsilon\int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s)\,ds\right\|_{L^1}\\
&\leqslant n\,\|\nabla^{k+1}\phi\|_\infty\sqrt{\tau}\,\sup_T\left\|\frac{1}{\sqrt{T}}\int_0^T f(Y_s)\,ds\right\|_{L^1}\\
&\leqslant C\sqrt{\tau}
\end{aligned}
$$

Proof of (c). The last part of this lemma is obtained by using the two arguments above and applying the Ascoli's theorem.

(b) Weak Limit. The family $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\mathbb{R}^+\times\mathbb{R}^n)$ by $\|\phi\|_{L^\infty}$. By the Banach–Alaoglu theorem, the family u_ε is relatively weakly*-compact in $L^\infty(\mathbb{R}^+\times\mathbb{R}^n)=(L^1(\mathbb{R}^+\times\mathbb{R}^n))'$. Let u be a limit point of this family and rename as usual, ϕ_ε, ψ_ε and u_ε the corresponding subfamilies, with u_ε converging to u. We have for any function χ integrable on $\mathbb{R}^+\times\mathbb{R}^n$,

$$
\lim_{\varepsilon\to 0}\int_{\mathbb{R}^+\times\mathbb{R}^n} u_\varepsilon(t,x)\,\chi(t,x)\,dt\,dx=\int_{\mathbb{R}^+\times\mathbb{R}^n} u(t,x)\,\chi(t,x)\,dt\,dx
$$

This holds in particular for a function χ belonging to $\mathscr{C}_K^\infty$. Hence, for any $\chi\in\mathscr{C}_K^\infty$, we define for all $\varepsilon>0$ and $\tau>0$ the quantity

$$
\begin{aligned}
\Omega_{\varepsilon,\tau}(\chi) &:= \left\langle\frac{u_\varepsilon(\cdot+\tau,\cdot)-u_\varepsilon(\cdot,\cdot)}{\tau},\chi\right\rangle\\
&=\int_{\mathbb{R}^+\times\mathbb{R}^n}\frac{u_\varepsilon(\cdot+\tau,\cdot)-u_\varepsilon(\cdot,\cdot)}{\tau}\,\chi(t,x)\,dt\,dx\\
&=\int_{\mathbb{R}^+\times\mathbb{R}^n}u_\varepsilon(t,x)\,\frac{\chi(t-\tau,x)-\chi(t,x)}{\tau}\,dt\,dx
\end{aligned}
$$

Since $(\chi(\cdot - \tau, x) - \chi(\cdot, \cdot)/\tau) \in \mathscr{C}_K^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, for all $\tau > 0$, $\lim_{\varepsilon \to 0} \Omega_{\varepsilon, \tau}$ exists and is equal to

$$\Omega_\tau(\chi) = \int_{\mathbb{R}^+ \times \mathbb{R}^n} u(t, x) \frac{\chi(t-\tau, x) - \chi(t, x)}{\tau} \, dt \, dx$$

Now letting $\tau \to 0$, yields

$$\begin{aligned} \Omega(\chi) := \lim_{\tau \to 0} \Omega_\tau(\chi) &= -\int_{\mathbb{R}^+ \times \mathbb{R}^n} u(t, x) \frac{d}{dt} \chi(t, x) \, dt \, dx \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}^n} \frac{d}{dt} u(t, x) \, \chi(t, x) \, dt \, dx \end{aligned}$$

Starting with Eq. (2.13), using Proposition 4.1, the first term of the right hand side of Eq. (2.13) tends to 0 as $\varepsilon \to 0$. On the other hand,

$$\begin{aligned} \lim_{\varepsilon \to 0} \Bigg[& \tfrac{1}{2} \left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s) \, ds \right) : \left(\varepsilon \int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s) \, ds \right)^{\otimes 2} \right\rangle \\ & - \tfrac{1}{2} \left\langle \nabla_x^2 \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s) \, ds \right) \right\rangle : \left\langle \left(\varepsilon \int_{t/\varepsilon^2}^{(t+\tau)/\varepsilon^2} f(Y_s) \, ds \right)^{\otimes 2} \right\rangle \Bigg] = 0 \end{aligned}$$

By the dominated convergence Theorem, it is not hard to see that $u_\varepsilon \in \mathscr{C}^2(\mathbb{R}^n)$; since $\nabla^2 \phi$ is bounded, $\nabla_x^2 u_\varepsilon(t, x) = \langle \nabla_x^2 \psi_\varepsilon(t, x, \cdot) \rangle$. Hence, for any function $\chi \in \mathscr{C}_K^\infty$, we have

$$\Omega_{\varepsilon, \tau}(\chi) = \frac{1}{2} \left\langle \nabla_x^2 u_\varepsilon : \left\langle \left(\frac{\varepsilon}{\sqrt{\tau}} \int_0^{\tau/\varepsilon^2} f(Y_s(y) \, ds \right)^{\otimes 2} \right\rangle + r_{\varepsilon, \tau, t}, \chi \right\rangle$$

Thus

$$\begin{aligned} & \limsup_{\varepsilon \to 0} \left| \Omega_{\varepsilon, \tau}(\chi) - \int_{\mathbb{R}^+ \times \mathbb{R}^n} \nabla_x^2 u_\varepsilon(t, x) : \tfrac{1}{2} \kappa^2(\tau/\varepsilon^2) \, \chi(t, x) \, dt \, dx \right| \\ & \quad \leqslant \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^+ \times \mathbb{R}^n} |r_{\varepsilon, \tau, \cdot} \cdot \chi(t, x)| \, dt \, dx \\ & \quad \leqslant \limsup_{\varepsilon \to 0} \sup_t |r_{\varepsilon, \tau, \cdot}| \cdot \|\chi\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^n)} \\ & \quad \leqslant O(\tau)^{1/2} \end{aligned}$$

Moreover, since $\chi \in \mathscr{C}_K^\infty$, taking into account Eq. (4.46), yields

$$\int_{\mathbb{R}^+ \times \mathbb{R}^n} \chi(t, x) \frac{(u(t+\tau, x) - u(t, x)}{\tau} \, dt \, dx$$

$$= \int_{\mathbb{R}^+ \times \mathbb{R}^n} \left(\frac{1}{2} \kappa^2 : \nabla_x^2 u(t, x) \right) \chi(t, x) \, dt \, dx + O(\tau)^{1/2}$$

Next, taking the limit as $\tau \to 0$ to get rid of the term of $O(\tau)^{1/2}$; we deduce that u is a solution of the initial value problem for the diffusion equation

$$\frac{du}{dt} = \frac{1}{2} \kappa^2 : \nabla_x^2 u, \qquad u(0, x) = \phi(x)$$

The solution of the Cauchy problem defined in the Theorem 2.1 is uniquely defined and therefore not only a subsequence, but the complete family

$$\langle \varphi_\varepsilon(t, x, y) \rangle \text{ or } u_\varepsilon(t, x) = \langle \psi_\varepsilon(t, x, y) \rangle = \left\langle \phi \left(x + \varepsilon \int_0^{t/\varepsilon^2} f(Y_s(y)) \, ds \right) \right\rangle$$

converges in $C^0([0, \tau], w^* - L^\infty(\mathbb{R}^+ \times \mathbb{R}^n))$ to $u(t, x)$.

We now complete the proof of Theorem 2.1. We shall now prove that the limit points correspond to the functions of class $\mathscr{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n)$. In order to do so, we consider a limit point u of $(u_\varepsilon)_{\varepsilon > 0}$ in $\mathscr{C}^{0,2}(\mathbb{R}^+ \times \mathbb{R}^n)$ and for the weak topology on L^∞. In the sense of distributions, we have:

$$\frac{d}{dt} u = \frac{1}{2} \kappa^2 : \nabla_x^2 u(t, x)$$

We deduce that the distribution $(d/dt) \, u$ corresponds to a continuous function w on $\mathbb{R}^+ \times \mathbb{R}^n$. Let v be the function defined by

$$v(t, x) = \phi(x) + \int_0^t w(s, x) \, ds$$

Note that $v \in \mathscr{C}^{1,0}(\mathbb{R}^+ \times \mathbb{R}^n)$. Put $\Gamma := \{h'; h \in \mathscr{C}_K^\infty(\mathbb{R}^+)\}$. It is not hard to see that, for any $g \in \Gamma$, and $x \in \mathbb{R}^n$,

$$\int_0^{+\infty} (u - v)(t, x) \, g(t) \, dt = 0$$

and that $\Gamma = \{g \in \mathscr{C}_K^\infty(\mathbb{R}^+) : \int_{\mathbb{R}} g(t)\, dt = 0\}$. Let ϕ_0 be a function defined in $\mathscr{C}_K^\infty(\mathbb{R}^+)$ such that $\int_{\mathbb{R}} \phi_0(t)\, dt = 1$. Then, for any function f in $\mathscr{C}_K^\infty(\mathbb{R}^+)$, the function $f - \phi_0 \int_0^{+\infty} \phi_0(s)\, ds$ belongs to Γ, and so for any $x \in \mathbb{R}^n$, we have

$$\int_0^{+\infty} (u-v)(t,x)\, f(t)\, dt = \int_0^{+\infty} (u-v)(t,x)\, \phi_0(t)\, dt \int_0^{+\infty} f(s)\, ds$$

Using the continuity of the function $z := (u-v)$, we finally get

$$z(t,x) = \int_0^{+\infty} (u-v)(t,x)\, \phi_0(t)\, dt = z(0,x) = 0$$

Hence, we deduce that u coincide with v. Thus, the limit points for uniform convergence on any compact $[0,\tau] \times X$ and the weak* convergence on $L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ are the solutions u of class $\mathscr{C}^{1,2}$ of the heat equation (2.11), with the initial data $u(0,\cdot) \equiv \phi$. The existence and uniqueness of such a solution is a classical result (uniqueness is given by the Maximum principle). The proof of Theorem 2.1 is now complete. ∎

APPENDIX

It is illuminating to see how the eigenvalues of P-F operator allows us to obtain Kuzmin's theorem in Section III. An essential point was that, to find an appropriate setting for a formulation of Kuzmin's theorem, we need to investigate the P-F operator $\mathscr{L}_\beta^{(0)}$ which possess a dominant eigenvalue λ_1 and a subdominant eigenvalue λ_2 satisfying $|\lambda_1| > |\lambda_2|$. This can be stated as

Lemma 5.1. The operator $\mathscr{L}_\beta^{(0)}: E_\infty(D) \to E_\infty(D)$ defined by Eq. (3.4) (with $s=0$) has a simple positive dominant eigenvalue $\lambda_1 = 1$ and a simple negative subdominant eigenvalue λ_2 with $|\lambda_i| < -\lambda_2 < \lambda_1 = 1$ for all $i \geqslant 3$.

Proof. The proof is the same as in ref. 12; for sake of completeness we present the proof in detail. It will be convenient to introduce the following notation:

$E_{1,\infty}^{\mathbb{R}}(D)$ the real subspace of $E_{1,\infty}(D)$ of all f that takes real values on $D_{\mathbb{R}} = \mathbb{R} \cap D$;

$\tilde{E}_{1,\infty}^{\perp}(D) = \{f \in E_{1,\infty} : \int_0^1 f(x)\, dx = 0\}$; $E_{1,\infty}^{\perp}(D) = \{\hat{f} \in E_{1,\infty} : \int_0^1 \hat{f}(x)\, h_1(x)\, dx = 0\}$.

$E^{\perp, \mathbb{R}}_{1, \infty}(D)$ is the real subspace of all $\hat{f} \in E^{\perp, \mathbb{R}}_{1, \infty}(D)$ which take real values on $D_{\mathbb{R}}$, and in this space the cone C is defined as $C = \{ \hat{f} \in E^{\perp, \mathbb{R}}_{1, \infty}(D) : \hat{f}' \geqslant 0$ on $D_{\mathbb{R}} \}$.

Next set $V: E_{1, \infty}(D) \to E_{1, \infty}(D)$, $Vf(z) = \sum_{n=1}^{\infty} (z+1)/((z+n)(z+n+1))\, f(1/(z+n))$.

The problem is to determine the eigenvalue λ_2. It can be obtained by the minimax principle. We have:

$$\min_{\hat{f} \in \mathring{C}} \max_{x \in \bar{D}_{\mathbb{R}}} \frac{(V\hat{f})'(x)}{\hat{f}'(x)} = \lambda_2 = \max_{\hat{f} \in \mathring{C}} \min_{x \in \bar{D}_{\mathbb{R}}} \frac{(V\hat{f})'(x)}{\hat{f}'(x)} \tag{5.1}$$

From this we get rigorous upper and lower bounds for the eigenvalues λ_2:

$$\min_{x \in \bar{D}_{\mathbb{R}}} \frac{(V\hat{f})'(x)}{\hat{f}'(x)} \leqslant \lambda_2 \leqslant \max_{x \in \bar{D}_{\mathbb{R}}} \frac{(V\hat{f})'(x)}{\hat{f}'(x)} \tag{5.2}$$

where $\hat{f}$ is any element in the interior $\mathring{C}$ of the cone C. The above minimax principle should to be compared with a completely analogous one for the highest eigenvalue λ_1 of the operator $\mathscr{L}^{(0)}_{\beta}$:

$$\max_{\hat{f} \in \mathring{K}_+} \min_{x \in \bar{D}_{\mathbb{R}}} \frac{\mathscr{L}^{(0)}_{\beta} f(x)}{f(x)} = \lambda_1 = \min_{\hat{f} \in \mathring{K}_+} \max_{x \in \bar{D}_{\mathbb{R}}} \frac{\mathscr{L}^{(0)}_{\beta} f(x)}{f(x)} \tag{5.3}$$

K_+ is the cone $K_+ = \{ f \in E^{\mathbb{R}}_{1, \infty}(D) : f(x) \geqslant 0$ on $D_{\mathbb{R}} \}$ in the real Banach space $E^{\mathbb{R}}_{1, \infty}(D)$. However as $\int_0^1 \mathscr{L}^{(0)}_{\beta} f(x)\, dx = \int_0^1 f(x)\, dx$, this formula is not interesting, λ_1 must be 1 anyhow. This gives for instance $\zeta[2; \frac{7}{2}] \leqslant \lambda_1 \leqslant \zeta[2, \frac{1}{2}]$ where $\zeta(z, q) = \sum_{i=0}^{\infty} 1/(q+i)^z$ is the Hurwitz function. The proof of lemma is complete now. ∎

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REFERENCES

1. D. V. Anosov, Averaging in systems of ODEs with rapidly oscillating solutions, *Izv. Akad. Nauk. SSSR* **24**:721–742 (1960) (untranslated).

2. V. I. Arnold, Conditions for the applicability, and estimate of the error, of an averaging method for systems which pass through states of resonance in the course of their evolution, *Dokl. Akad. Nauk.* **161**(1):331–334 (1965) [English translation: *Soviet Math. Doklady* **6**(2):581–585 (1965)].
3. P. Arnoux, Le Codage du Flot Géodésique sur la Surface Modulaire, *L'enseignement Mathématique* **40**:29–48 (1994).
4. C. Bardos, J.-F. Colonna, and F. Golse, Diffusion approximation and hyperbolic automorphisms of torus, *Physica D*; *Nonlinear Phenomene* **104**(1):32–60 (1997).
5. R. Dautray, *Méthodes Probabilistes pour les Équations de la Physique* (Masson, 1989).
6. J.-P. Conze and S. Le Borgne, Méthode de martingales et flot géodésique sur une surface de courbure négative, prépublication 99-16 de IRMAR (Université de Rennes).
7. C. Dogbé, Examples of approximation by diffusion for kinetic equations, Ph.D. thesis (Univ. Paris VII, 1998).
8. H. Scott Dumas and F. Golse, The averaging method for pertubation of mixing flows, *Ergod. Th. Dynam. Sys.* **17**, Part 6 (1997).
9. Y. Guivarc'h and Y. Le Jan, Asymptotic winding of the geodesic flow on modular surfaces and continuous fractions, *Ann. Sc. Ec. Norm. Sup.* **26**:23–50 (1993).
10. Y. Katznelson, Ergodic automorphisms of T^n are Bernoulli shifts, *Israel J. of Math.* **10**:186–195 (1971).
11. R. Kuzmin, A problem of Gauss, *Dokl. Akad. Nauk SSSR A* 375–380 (1928); in *Atti del Congresso Intern. dei Matematici (Bologna)* **6**:83–89 (1928).
12. D. Mayer, Ergodic theory, *Symbolic Dynamics and Hyperbolic Spaces* (Oxford Sc. Pub, 1991).
13. V. B. Minasjan, Some results of Gibbs local-equilibrium distributions, *Dokl. Akad. Nauk SSSR* **242**(5):1004–1007 (1978) [in Russian].
14. C. Series. Symbolic dynamics for geodesic flows, *Acta Math.* **146**:103–128 (1981).
15. M. Ratner, The central limit theorem for geodesic flows on n-dimensional manifolds of negative curvature, *Israel J. Math.* **16**:181–197 (1973).